## 2.3: Interpretations of the Derivative

For a function $f$, we have seen the derivative $f'$ represented as the instantaneous rate of change. That is to say that $f'(x)$ is the limit as we take the average rate of change in smaller and smaller intervals containing $x$. For this reason you can think of

$$f'(x) \approx \frac{\Delta y}{\Delta x}.$$

However, there is another notation for derivatives, introduced by the German mathematician Leibniz. When $y = f(x)$ he writes

$$f'(x) = \frac{dy}{dx}.$$

With this notation it is easy to see that the derivative is represented as

$$\frac{\text{Difference in } y\text{-values}}{\text{Difference in } x\text{-values}}.$$

In order to evaluate the Leibniz notation at a point, such as $f'(2) = 4$ say the derivative of $f$ at 2 is 4, we write

$$\left.\frac{dy}{dx}\right|_{x=2} = 4.$$

### Using Units to Interpret the Derivative

**Example 1:** The cost $C$ (in dollars) of building a house $A$ square feet in area is given by the function $C = f(A)$. What are the units and the practical interpretation of the function $f'(A)$?

units of $f'(A) =$ dollars/square foot

In general, the units for the derivative function are the same as the units for average rate of change.

1. The units of the derivative of a function are the units of the dependent variable divided by the units of the independent variable. In other words, the units of $dA/dB$ are the units of $A$ divided by the units of $B$.
2. If the derivative of a function is not changing rapidly near a point, then the derivative is approximately equal to the change in the function when the independent variable increases by 1 unit.

**Example 2:** The cost of extracting $T$ tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say $f'(2000) = 100$?

Then cost of removing 2001 tons of ore is approximately \$100 more than removing 2000 to

**Example 3:** If $q = f(p)$ gives the number of thousands of tons of zinc produced when the price is $p$ dollars per ton, then what are the units and the meaning of

$$\left.\frac{dq}{dp}\right|_{p=900} = 0.2?$$

The ~~cost of~~ number of thousands of tons of zinc produced at \$901 is approximately 0.2 greater than thousand of tons produced at \$900.

**Example 4:** The time, $L$ (in hours), that a drug stays in a person's system is a function of the quantity administered, $q$, in mg, so $L = f(q)$.

(a) Interpret the statement $f(10) = 6$. Give units for the numbers 10 and 6.

(b) Write the derivative of the function $L = f(q)$ in Leibniz notation. If $f'(10) = 0.5$, what are the units of 0.5?

(c) Interpret the statement $f'(10) = 0.5$ in terms of dose and duration.

(a) If 10mg is administered it will take 6 hours to leave your system

(b) $\left.\frac{dL}{dq}\right|_{q=10} = 0.5 \text{ hours/mg.}$

**Example 5:** If the velocity of a body at time $t$ seconds is measured in meters/sec, what are the units of the acceleration of the body?

$\text{meters}/\text{sec}^2$

## Tangent Line Approximation: Local Linearity

Recall that the derivative tells us how fast the value of a function is changing. So if a function is relatively tame near a point we can use the derivative to estimate values of the function at nearby points.

**Example 6:** Suppose that $f(t)$ is a function with $f(25) = 3.6$ and $f'(25) = -0.2$. Estimate $f(26)$ and $f(30)$.

$\boxed{\Delta x f'(t) = \Delta y}$

For $f(26)$, $\Delta y = -0.2 = 1 \cdot (-0.2)$

so $f(26) \approx 3.4$.

For $f(30)$, $\Delta y = 5 \cdot (-0.2) = -1$ so $f(30) \approx 2.6$.

**Definition:** If $y = f(x)$ and $\Delta x$ is near 0, then $\Delta y \approx f'(x)\Delta x$. For $x$ near $a$, we have $\Delta y = f(x) - f(a)$, so

$$f(x) \approx f(a) + f'(a)\Delta x$$

is called the **tangent line approximation** of $f(x)$.

**Example 7:** For a function $f(x)$, we know that $f(20) = 68$ and $f'(20) = -3$. Estimate $f(21)$, $f(19)$ and $f(25)$.

$f(21) \approx f(20) + f'(20) \cdot 1 = 65$

$f(19) \approx f(20) + f'(20) \cdot (-1) = 71$

$f(25) \approx f(20) + f'(20) \cdot 5 = 53$

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Recall that for relative change of a function we look at the rate of change as a fraction of the original value of the function. We can define relative change analagously with the derivative function.

**Definition:** The **relative rate of change** of $y = f(t)$ at $t = a$ is defined to be

$$\text{Relative rate of change of } y \text{ at } a = \frac{dy/dt}{y} = \frac{f'(a)}{f(a)}.$$

**Example 8:** Annual world soybean production, $W = f(t)$, in million tons, is a function of $t$ years since the start of 2000.

(a) Interpret the statements $f(8) = 253$ and $f'(8) = 17$ in terms of soybean production.

(b) Calculate the relative rate of change of $W$ at $t = 8$; interpret it in terms of soybean production.

(a) In 2008, 253 million tons of soybean produced

Around 2008, production was increasing by 17 million tons a year.

(b) $\frac{f'(8)}{f(8)} = \frac{17}{253} = 6.7\%$

**Example 9:** Solar photovoltaic (PV) cells are the world's fastest growing energy source. At time $t$ in years since 2005, peak PV energy-generating capacity worldwide was approximately $E = 4.6e^{0.43t}$ gigawatts. Estimate the relative rate of change of PV energy-generating capacity in 2015 using this model and

(a) $\Delta t = 1$ (b) $\Delta t = 0.1$ (c) $\Delta t = 0.01$

(a) $\frac{f'(10)}{f(10)} \approx \frac{1}{f(10)} \cdot \frac{f(11)-f(10)}{1} = 0.537 = 53.7\%$ per year

(b) $\frac{f'(10)}{f(10)} \approx \frac{1}{f(10)} \cdot \frac{f(10.1)-f(10)}{0.1} = 0.439 = 43.9\%$ per year

(c) $\frac{f'(10)}{f(10)} \approx \frac{1}{f(10)} \cdot \frac{f(10.01)-f(10)}{0.01} = 0.431 = 43.1\%$ per year

These are approaching 43% which we know is the relative change.

**Examply 10:** In April 2009, the US Bureau of Economic Analysis announced that the US gross domestic product (GDP) was decreasing at an annual rate of 6.1%. The GDP of the US at that time was 13.84 trillion dollars. Calculate the annual rate of change of the US GDP in April 2009.

$\frac{f'(2009)}{f(2009)} = \overset{-0.061}{\cancel{1+0.061 = 0.939}}$

So $f'(2009) = \overset{-0.061(13.84) = -0.84 \text{ trillion \$/yr}}{\cancel{1+0.061(13.84)}}$ $\cancel{0.939(13.84)}$

Thus $f(2010) \approx 12.996$ trillion dollars